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UNDAMPED MASS-SPRING CASE GENERATION

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ABSTRACT

The mass-spring system is subject to natural laws of gravity and the reaction to the spring wire and the surrounding, so the swing is affected by these factors that are in the form of forces, and therefore the continued swing of the wire depends on the effect of these forces affecting the body attached to the spring wire, but it is possible to influence one way or another with an external force to quench this power to keep the wire moving continuously. If this is achieved, this phenomenon can be exploited in various practical fields such as energy production, for example.

This paper will be concerned with how to influence with an external force that ensures the continuity of the spring wire movement. Since the responsibility for the motion of this system is a second-order differential equation, the damping coefficient listed within the wire movement equation will be dealt with the addition of an external force.

KEYWORD: Mass-Spring, Damped Mass-Spring, Second Order ODE, Harmonic Oscillation, Stability of 2^{nd} order ODE.

INTRODUCTION

In physical systems, damping is produced by processes that dissipate the energy stored in the oscillation, such as in mechanical systems, resistance in electronic oscillators, and absorption and scattering of light in optical oscillators, the force supplied by a shock absorber in a car or a bicycle. The damping force is proportional to the velocity of the mass and acts in the direction opposite to the motion [9]. Also, it is used in modeling on chemical processes, control theory, electromagnetism, thermodynamics, and many other problems of physics, engineering, and the various works cited therein [6].Study of mass-spring-damper system models provides insight into a wide range of practical engineering problems [4, 9]**.**

One way to get a better approximation to reality is to assume a damping force. The exact law for this force is not known since it depends on so many variable factors, but for small speed, it is found, experimentally, that the damping force is approximately proportional to the instantaneous speed of the weight on the spring [6].

The damping force magnitude is given by

cy',

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where $c > 0$ is proportionality damping constant, the damping force opposes the motion which acts in the opposite of the motion, you can say, so the sign of the damping force has a negative of the velocity $y'(t)$, damping is an influence within or upon an oscillatory system that has the effect of reducing, restricting or preventing its oscillations[9].

The description of the motion of a spring-mass system begins with Newton Second Law of the motion:

$F = ma,$ (1)

where$F$ is a force, a is an acceleration and m is a mass of the object [8].

Actually, there are three primaries of forces acting on the mass [1]. The first is the gravity which pulled the mass downward with a force mg,

$wight\ force = mg.$ (2).

Next, the spring exerts a restoring force, it is stretched or compressed, and Hook's Law states that, this force is proportional to the displacement from the spring's natural length

$spring\ force = -k(y + \delta y),$ (3)

for some $k > 0$ (spring constant). The third force on the system is damping force that resists the motion. The damping force depends on the velocity of motion of the system; more damping for faster motion

$damping\ force = -cv,$ (4)

where$c > 0$ isa damped constant, $v = y'$ is the velocity.

From (1), (2), (3) and (4)

$F = ma = my'' = mg - k[y + \delta y] - cy'(t),$

$= mg - k[y + \delta y] - cy',$

$my'' + cy' + ky = mg - k\delta y.$

then

$my''(t) + cy'(t) + ky(t) = f(t),$ (5)

which is a second-order differential equation; Hook's law, where m the mass of the body, c is the damped constant and k is the spring constant, all of them are positive constants [7].

If $f(t) = 0$ then the equation has a homogeneous form

$my''(t) + cy'(t) + ky(t) = 0.$ (6)

The stability of the second-order differential equation:

The system is said to be a stableif for every $\epsilon > 0$there exists a $\delta > 0$such that $\|x_0(t)\| < \delta$implies$\|f(x_0(t))\| < \epsilon \forall t \in \mathbb{R}^+$. The origin is attractiveif it is contained in some open neighborhood [3].

The spring-mass motion, as described above, is an application of the second-order of ordinary differential equation (ODE), so the characteristic equation which governs the situation has a form

$$a_0 y'' + a_1 y' + a_2 y = g(t). \qquad (7)$$

The equation is nonhomogeneous, t is the time and a_i $(i = 0, 1, 2)$ are constants.

The general solution is

$$y = C_1 y_1 + C_2 y_2 + y_p, \qquad (8)$$

wherey_p is a particular solution of (7), and C_1, C_2 are constants.

The homogeneous form of the equation (7) is

$$a_0 y'' + a_1 y' + a_2 y = 0, \qquad (9)$$

The solution $y_c = C_1 y_1 + C_2 y_2$ (complementary solution) is a general solution to the associated homogeneous equation (9). The values of the constants C_1 and C_2 could be determined using initial conditions.

The general solution molded by (8) is a stable if it leads to, for every choice of C_1, C_2 such that

$$C_1 y_1 + C_2 y_2 \to 0 \text{as} t \to \infty.$$

If the system is a stable then the two parts of the solution (8); the particular solutiony_p and the complementary solution $y_c$are prescribed as steady-state solution and transient respectively.

The second-order differential equation can be transferred into a system of first-order differential equations;

let

$$x_1 = y,$$

$$x_2 = y',$$

then

$$x_1' = y' = x_2.$$

Apply this process to the equation (6)

$$x_2' = y'' = -\frac{k}{m} x_1 - \frac{c}{m} x_2.$$

Then the system will be in matrix form as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{c}{m} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

$$\boldsymbol{X}' = \boldsymbol{AX}, \qquad (10)$$

$$\det(A - \lambda I) = 0,$$

$$(\lambda I - A) = \begin{pmatrix} -\lambda & 1 \\ -\frac{k}{m} & -\frac{c}{m} - \lambda \end{pmatrix} = 0,$$

$$\lambda_{1,2} = \left[-\frac{c}{m} \pm \sqrt{\frac{c^2}{m^2} - 4\frac{k}{m}} \right] / 2,$$

$$= [-c \pm \sqrt{c^2 - 4mk}]/2m, \qquad (11)$$

Since, the values of m, c and k are positive, so the first part of (11) will be always negative. The second part will depend on the sign of

$(c^2 - 4mk)$, it has the following cases:

1) $c^2 - 4mk > 0$, the roots $\lambda_{1,2}$ have real values.
2) $c^2 - 4mk < 0$, the roots $\lambda_{1,2}$ have complex values.
3) $c^2 - 4mk = 0$,the roots $\lambda_{1,2}$ have the same real values.

The possibility roots of equation (11) and the solutions of (6) are shown in the following table1:

Root	***Solution***	***Condition***
$\lambda_1 \neq \lambda_2$	$C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$	$\lambda_1, \lambda_2 < 0$
$\lambda_1 = \lambda_2 = \lambda$	$e^{\lambda t}(C_1 + tC_2)$	$\lambda < 0$
$a \pm ib$	$e^{at}(c_1 \cos bt + c_2 \sin bt)$	$a < 0$

Table 1: The possibilities root of characteristic equation (11) and the associated solutions of equation (6)

and the solution is

$$y = C_1 y_1 + C_2 y_2.$$

With initial values, the constants C_1 and C_2 could be determined as mention above.

Stability conditions of the second-order differential equation:

The stability of a system means that its tendency to seek a condition of state equilibrium after it has been disturbed. When a system in stable equilibrium is given a small displacement, it is said to be a stable if it returns to the equilibrium point, and it is said to be an unstable if it moves further more away.

Definition:

The system will be a stable if the terms of the equation (7) does not depend on the initial condition, and it will be a stable if and only if C_1 and C_2 make the solution

$C_1y_1 + C_2y_2 \to 0$,as $t \to \infty$.

We wonder about the conditions for normal differential equation (7) to be a stable. By studying the equations (7), (8) and (9), since the stability is related to the behaviors of the solution of the homogeneous equation (9), where the limit $g(t)$ in the equation (7) does not play a role in determining whether it is a stable or not. There are two independent linear solutions for the equation (9), one of which is assumed to be in the exponential form:

$y = e^{rt} \implies y' = re^{rt} \implies y" = r^2e^{rt}$.

By the substation with y and its derivatives in the equation (9), we get,

$(a_0r^2 + a_1r + a_2)e^{rt} = 0$,

becausee$^{rt} \neq 0$, then

$a_0r^2 + a_1r + a_2 = 0$,

which has two roots;

$r_{1,2} = \frac{-a_1 \pm \sqrt{(a_1^2 - 4a_0a_2)}}{2a_0}$.

These roots have three possibilities:

1) The roots are real if $a_1^2 - 4a_0a_2 > 0$, in this case the solutions e^{r_1t} and e^{r_2t} are independent, and the general solution has the form:

$y = C_1e^{r_1t} + C_2e^{r_2t}$.

2) The roots are complex conjugates if $a_1^2 - 4a_0a_2 < 0$, $r_1 = a + ib$,$r_2 = a - ib$)$i = \sqrt{-1}$, and the solutions e^{r_1t} and e^{r_2t} are independent, and the general solution has the form:

$y = e^{at}[C_1 \, cos(bt) + iC_2 \, sin(bt)]$.

3) The roots are equal if $a_1^2 - 4a_0a_2 = 0$, $r_1 = r_2 = \frac{-a_1}{2a_0}$, one of them is $exp(\frac{-a_1}{2a_0}t)$.

Stability Cases:

Stability cases are shown in the table (1). From the table, it could be seen the following results:

Case 1: if $r_1, r_2 < 0$, then the solution goes to zero as $t \to \infty$, that is meant the solution is a stable. But if $r_1 \geq 0$ the solution e^{r_1t} tends to infinity that means the solution is unstable.

Case 2: the situation will be the same as the case 1, for $r < 0 \implies lim_{t\to\infty}(te^{rt}) = 0$.

Case 3: if $a < 0$, then $lim_{t\to\infty} e^{at} \, cos(bt) = 0$ and $lim_{t\to\infty} e^{at} \, sin(bt) = 0$.

The three cases of the stability scale for the second-order differential equation (7) in terms of roots are summarized in one sentence:

The second-order differential equation is a stable if and only if all the roots of the characteristic equation have real negative part.

As for the measure of stability of the second-order differential equation (7) by coefficients form; it is a stable if and only if $a_0, a_1, a_2 > 0$ [1].

The motion of the linearized mass-spring:

The mass-spring has a vertical motion of body subjects to the second-order differential equation; Hook's law which numbered by (5) above:

$$my'' + cy' + ky = f(t). \qquad (12)$$

All the possibility motions of the body at the end of a spring (oscillation of a linearized spring) are determined when all solutions of

$$my'' + cy' + ky = 0; \qquad (13)$$

the complementary solutiony_c, which is known a free solution, andthe single particular solution y_p (forced solution) of (12) are known.

The equation (13) could be written as:

$$y'' + \frac{c}{m}y' + \frac{k}{m}y = 0. \qquad (14)$$

In the absence of a damping term$c$, the ratio k/m would be the square of the circular frequency of the solution, so we will write $\frac{k}{m} = \omega_n^2$with $\omega_n > 0$, and call ω_n the natural circular frequency of the system.

Rewrite the equation (14) as:

$$y'' + \left(\frac{c}{m}\right)y' + \omega_n^2 y = 0.$$

Critical damping occurs when the coefficient of y' is $2\omega_n$. The damping ratio ζ is the ratio of $\frac{c}{m}$ to the critical damping constant; $\zeta = (\frac{c}{m})/2\omega_n$, damping ratio. The equation becomes as:

$$y'' + 2\zeta\omega_n y' + \omega_n^2 y = 0. \qquad (15)$$

In general the characteristic polynomial is

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2,$$

and has roots

$$\lambda_{1,2} = -\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2 - \omega_n^2} = \omega_n(-\zeta \pm \sqrt{\zeta^2 - 1}),$$

which are real, over damped system if $|\zeta| \geq 1$, equal - critical damping, when $\zeta = \pm 1$, and unreal -under damped or lightly damped system when $0 < |\zeta| < 1$. Figure1.

If$|\zeta| < 1$, the roots are

$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_d,$$

where

$$\omega_d = \omega_n\sqrt{\zeta^2 - 1},$$

is the damped circular frequency of the system. These are complex numbers of magnitude ω_n and argument $\pm\theta$, where $-\zeta = \cos\theta$. Note that the presence of a damping term decreases the frequency of the solution to the undamped equation—the natural frequency ω_n - by the factor$\sqrt{\zeta^2 - 1}$. The general solution is

$y = Ae^{-\zeta\omega_n t}\cos(\omega_d t - \emptyset)$.

In undamped case; the equation (13) will be as

$my'' + ky = 0,$ (16)

which is a harmonic oscillatorand sinusoidal.

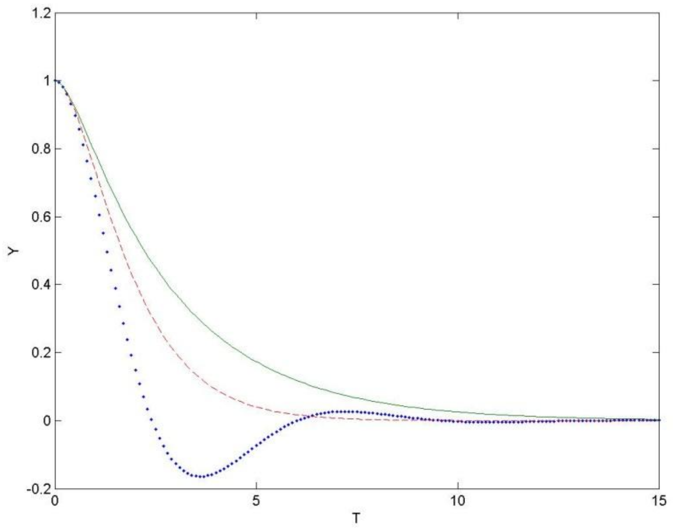

Figure 1: The solutions of the system; critical damping, dashed line,when $\zeta = \pm 1$, under damped or lightly damped system, pointed line, when $0 < |\zeta| < 1$, and over damping, thin line, $|\zeta| \geq 1$.

Undamped Case: Dynamics of Simple Harmonic Oscillation:

To get an undamped case, the system should be free from any effects (damping effects). Therefore, energy is purely transferred between the spring and the mass.

According to Hook's Law, when the mass pulled down by displacement, the spring produces a force F opposing the displacement, $F = -ky$, where k is the spring constant.

The force F, according to Newton's Second Law, can also be expressed, classically, as

$F = ma$,

thus

$my'' = -ky$,

$my'' + ky = 0$.

$y'' + \frac{k}{m}y = 0,$ (17)

which is a homogeneous linear second order differential equation, and its coefficients matrix has the form:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix},$$

and the eigenvalues

$\lambda_{1,2} = \pm i\sqrt{k/m}.$ (18)

the solution will be as

$y(t) = C\cos(\omega_n t) + D\sin(\omega_n t)$, (19)

$= A\cos(\omega_n t - \emptyset)$, (20)

where $A = \sqrt{C^2 + D^2}$, $\emptyset = \tan^{-1}(\frac{D}{C})$. This general solution is harmonic oscillation or simple harmonic motion, that means, the mass will oscillate indefinitely and the mass will freely oscillate at its natural frequency g, which in Hertz, is

$g = \frac{\omega_n}{2\pi} = \frac{1}{2\pi}\sqrt{(k/m)}$.

The constants C and D in the equation (19)could be found by using initial values for $t$and y,and get the solution. Each of constants in (20) carries a physical meaning of the motion;A is the amplitude (maximum displacement from the equilibrium position), $\omega = 2\pi g$ is the angular frequency, and $\emptyset$ is the phase.

With pure imaginary eigenvalues, the system is an unstable as discussed above. In this situation the system won't settle down at the equilibrium point and stays move forever.

By acting with an external term $f(t)$ on the equation (14) we get

$y'' + \frac{c}{m}y' + \frac{k}{m}y = f(t)$, (21)

where $f(t)$, by Newton Law, is ma, a mass multiplied by an acceleration y'' of the system.

If the mass is pulled down by displacement y_1 for example, then it is allowed by the spring force to return, the following displacement of the movement will be made by the mass is $y_2 < y_1$ because of the damping action, so if we could impact by a external force to let the mass does the same displacement as y_1 and so on for the rest of displacements in the following movements, the mass will oscillate indefinitely,

$y_2 < y_1$,

$y_1 - y_2 = \delta y$.

To omit δy, that is meant $y_2 = y_1$, we need some action which removes the displacement δy, Figure 2.

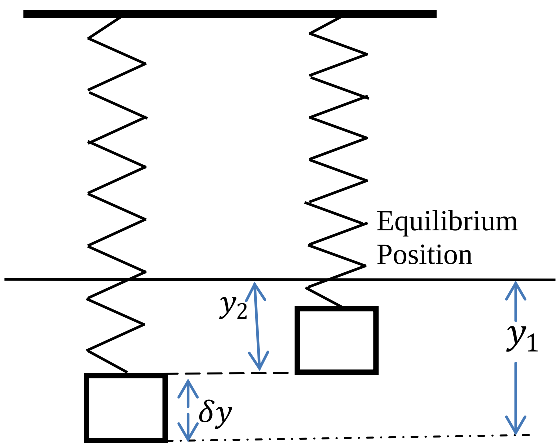

Figure 2:The difference, δy, between the first displacement y_1 and the second displacementy_2

By derivative the equation (20) to get the velocity $v = y'$;

$v = y' = -A\omega_n \, sin(\omega_n t - \emptyset)$, (22)

and the acceleration

$a = -A(\omega_n)^2 cos\,(\omega_n t - \emptyset)$. (23)

Mathematically, it could get this situation by adding a term $f(t) = ma$ to the right-hand side of equation (14) to get an undamped case (21),

$$f(t) = -m[A(\omega_n)^2 cos\,(\omega_n t - \emptyset)].$$

Let us consider the following example.

Consider the governing equation of the spring-mass system

$2y'' + 2y' + 5y = f(t), \quad y(0.5) = 1,\ y'(0.5) = 0.$

The example will be discussed in three cases, and Figure3 shows the behavior of the solutions.

Firstly, $f(t) = 0$, a homogeneous equation. The complementary solution is

$y_c = e^{-0.5x}(0.65 \cos\frac{3}{2}t + 1.19 \sin\frac{3}{2}t)$. (24)

Secondly, the equation has form $2y'' + 5y = 0$, which is an undamped case as discussed above, the solution has the form

$y = 0.69 \cos\sqrt{5/2}t - 0.71 \sin\sqrt{5/2t})$, (25)

the velocity v is

$v = y' = -1.12 \cos\sqrt{5/2}t - 1.09 \sin\sqrt{5/2}t$,

and the acceleration a is

$$a = y'' = -1.72 \cos\sqrt{5/2}t + 1.78 \sin\sqrt{5/2}t$$

Next, nonhomogeneous case, by using the Newton Law, $F = ma$, with the previous m and a:

$f(t) = ma = 2(-1.72 \cos\sqrt{5/2}t + 1.78 \sin\sqrt{5/2t})$,

the general solution has the form

$y = e^{-0.5x}\left(0.65 \cos\frac{3}{2}t + 1.19 \sin\frac{3}{2}t\right) + 0.708 \cos\sqrt{5/2}t - 0.688 \sin\sqrt{5/2t})$,

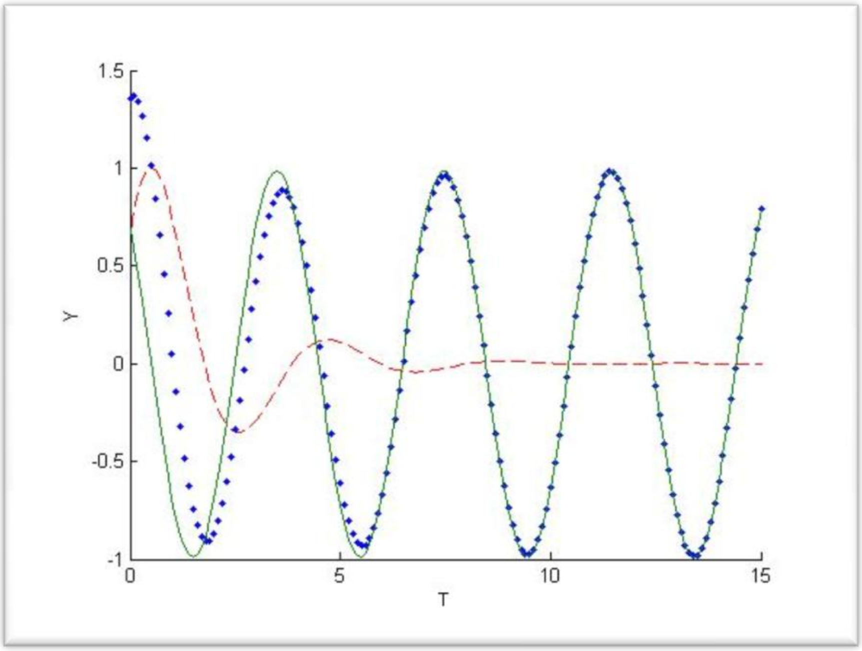

Figure 3: three cases of the solutions; solution of the homogeneous equation which has a damped solution, dashed line, the solution of undamped case, pointed line, and the solution of the nonhomogeneous equation with $f(t) \neq 0$, free line.

Practically, suppose there is such a system with damping constant, to omit the damping case, we need to add an external force to get a continuouslymovement. Such force may occur, for example, when the support holding the spring is moved up and down in a prescribed manner such as in periodic motion, or when the weight is given a little push every time it reaches the lowest position, but we don't know the values of the force, and where in the system, and when should be added.

The situation could be discovered by a couple of simple measurements of the system response.

CONCLUSION:

Through the foregoing, it was established that the mass-spring is related to various forces affecting the suspended body, and among these forces is what is known as the damping force that works to quench the swing to reach a state of stability, and the mass-spring system is an application of a second-order differential equation with constant coefficients, solving this equation leads to finding the eigenvalues of a system of first-order differential equations that have several states and to obtain the continuity of the mass-spring oscillation i.e. its instability must be the values of the eigenvalues have an imaginary incision only and this occurs if the suppression coefficient is which is a positive value. Practically, it cannot be ignored, so an external force has been added which in turn works to dampen the damping coefficient and keep the wire in a continuous state of oscillation. This external force is deduced from the special case where the differential equation is free from the suppression limit

In this way, a situation was obtained in which the system was unstable and in a continuous state of oscillation. This phenomenon can be used in practice obtaining kinetic energy if the system is well-built

Future work:

The spring wire can be studied in vitro to infer the non-damped state by using laboratory measurements of the mass-spring oscillating and the displacement and an external force that is used to get an undamped case, it could find some facts for the case and generate an undamped situation.

REFERENCES

1. Dawkins P., *Differential Equations*, http://tutorial.math.lamar.edu, 2018.
2. Hovhannisyan G., *Asymptotic stability for second-order differential equation with complex coefficients,* Electronic Journal of Differential Equations, Vol. 2004(2004), No. 85, pp. 1–20.
3. Koditschek D. E., Narendra K. J., *The Stability of Second Order Quadratic Differential Equations*, IEEE Transactions on Automatic Control, Volume AC-27, Issue 4, 1982, 783-798.
4. Longoria R.G, *Modeling and Experimentation: Mass-Spring-Damper System Dynamics,* Department of Mechanical Engineering the University of Texas at Austin, 2014.
5. Mendoza LJ., Cruz-Orduna I., Laguna-Camacho GR., *Experimental evaluation of viscous damping coefficient in the fractional under damped oscillator,* Advances in Mechanical Engineering, 2016, Vol. 8(4) 1–12.
6. Murray R. S., *applied differential equations*, Prentice-Hall,Inc Englwood Cliffs, N.J. 07632, 1981, USA.
7. Selvam A. G, Janagaraj R., International Journal of Recent Trends in Engineering & Research (IJRTER) *Volume 04, Issue 06; June - 2018 [ISSN: 2455-1457*].
8. Smith R. T., Minton R. B., *Calculus: Early Transcendental Functions*, McGrow-Hill Companies Inc, 2007, USA.
9. Sunday J., James A. A., Odekunle M. R., Adesanya A. O, *Solutions to Free Undamped and Free Damped Motion Problems in Mass-Spring Systems, American Journal of Computational and Applied Mathematics*, 2016, 6(2): 82-91.